

# Isoperimetry and Rough Path Regularity

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## Abstract

Optimal sample path properties of stochastic processes often involve generalized Hölder- or variation norms. Following a classical result of Taylor, the exact variation of Brownian motion is measured in terms of  $\psi(x) \equiv x^2/\log \log(1/x)$  near  $0+$ . Such  $\psi$ -variation results extend to classes of processes with values in abstract metric spaces. (No Gaussian or Markovian properties are assumed.) To establish integrability properties of the  $\psi$ -variation we turn to a large class of Gaussian rough paths (e.g. Brownian motion and Lévy's area viewed as a process in a Lie group) and prove Gaussian integrability properties using Borell's inequality on abstract Wiener spaces. The interest in such results is that they are compatible with rough path theory and yield certain sharp regularity and integrability properties (for iterated Stratonovich integrals, for example) which would be difficult to obtain otherwise. At last,  $\psi$ -variation is identified as robust regularity property of solutions to (random) rough differential equations beyond semimartingales.

## 1 Introduction

Optimal sample path properties of stochastic processes often involve generalized Hölder- or variation norms. For Brownian motion these results are classical and known as *Lévy's modulus* and *Taylor's variation* regularity respectively. Given an arbitrary stochastic process  $X(\omega) : [0, 1] \rightarrow (E, d)$ , we give a criterion (condition 7 below) which implies (via Garsia-Rodemich-Rumsey) Gauss tails for a Lévy-type  $\varphi$ -modulus "norm"<sup>1</sup>. In the very same setting, we show that  $X$  has a.s. finite  $\psi$ -variation "norm" of Taylor-type. In general,  $X$  need not be of a  $\psi^{-1}$ -modulus and one needs careful probabilistic arguments, adapted from Taylor [18] to our setting in theorem 11; these arguments are ill-suited to extract any integrability of the  $\psi$ -variation norms.

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<sup>1</sup>There is no linear space here but the analogy to well-known (semi)norms is strong enough to convince us to use the word "norm".

If  $X$  is a real-valued Gaussian process and the function  $\psi : [0, \infty) \rightarrow [0, \infty)$  is reasonably behaved (in particular, convex) one deals with a genuine (semi)norm and a classical result of Fernique implies (Gaussian) integrability. A more interesting example - and the motivation of the present work - are  $\mathbb{R}^d$ -valued centered *Gaussian rough paths* i.e. processes enhanced with a stochastic area process (such as *Lévy's area* in the case of Brownian motion). We view processes enhanced with their stochastic area as processes with values in  $G^2(\mathbb{R}^d) \cong \mathbb{R}^d \oplus so(d)$ , the step-2 nilpotent group equipped with Carnot-Caratheodory (CC) metric. The resulting Hölder and variation spaces (which play a fundamental rôle in *rough path theory*) have norms involving stochastic areas and hence do *not* allow us to use Fernique's result. Integrability properties of Wiener-Itô chaos (e.g. [10, Thm 4.1]) allow to go a bit further but ultimately fail to deal with the non-linear structure of the Hölder and variation "norms" of rough paths. To wit, the fine regularity properties required in rough path theory rely crucially on the cancellations on the right-hand-side of (1) below<sup>2</sup>. We overcome this difficulties with Borell's isoperimetric inequality which leads us to a *generalized Fernique estimate* (Theorem 4). As it may well be useful in other situations, we state and prove it in its natural setting of abstract Wiener spaces.

Keeping the recalls on rough path theory to a minimum, we remind the reader that rough paths take values in nilpotent groups with path regularity tied to the degree of nilpotency. For instance, Brownian motion and Lévy's area have (w.r.t. CC metric) finite  $p = (2 + \varepsilon)$ -variation and so one has to work in the group of step  $[p] = 2$  nilpotency. The main result in rough path theory, due to T. Lyons, is that higher iterated integrals, stochastic integrals of 1-forms and the Itô map (i.e. the solution map to stochastic differential equations) all become continuous and deterministic functions of *Brownian motion and Lévy's area*. The study of  $\psi$ -variation is then natural from several points of view;

- (i) it allows us to establish the definite variational regularity of Gaussian rough paths;
- (ii)  $\psi$ -variation regularity is the key to optimal regularity results for the coefficients of differential equations driven by rough paths (forthcoming work by A.M.Davie, not discussed here);
- (iii) we shall see that rough path estimates, usually stated in  $p$ -variation, are valid in suitable  $\psi$ -variation.

It may be helpful to state some of the implications of this work without using too much language of rough path theory and with focus on the simplest

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<sup>2</sup>With focus on Hölder regularity, a "linear" brute-force approach using integrability of Banach-space valued Wiener-Itô chaos leads to a Gauss tail of

$$\sup_{0 \leq s < t \leq 1} \frac{|A_{s,t}|^{1/2}}{|t-s|^\alpha}$$

for  $\alpha < 1/4$ , compare with the discussion preceeding (1). In order to apply rough path theory, however, it is crucial to take  $\alpha > 1/3$  so that  $[p] = 2$  with  $p = 1/\alpha$  and  $\alpha \in (1/3, 1/2)$ .

possible Gaussian rough path: Brownian motion and Lévy area. For instance, we have novel regularity and integrability properties of Lévy area increments defined as

$$A_{s,t} = \frac{1}{2} \left( \int_s^t (B_u - B_s) d\tilde{B}_u - \int_s^t (\tilde{B}_u - \tilde{B}_s) dB_u \right)$$

where  $(B, \tilde{B})$  is a 2-dimensional standard Brownian motion. The regularity result in the following theorem 1 below must not be confused with the essentially trivial statement that  $t \mapsto A_{0,t}$  has a.s. finite  $\psi$ -variation<sup>3</sup>. The situation is analogue to the subtle  $|A_{s,t}|^{1/2} \sim |t-s|^{1/2-\varepsilon}$  versus the simple  $|A_t - A_s| \sim |t-s|^{1/2-\varepsilon}$ , based on the cancellation taking place in the right hand side of

$$A_{s,t} = A_t - A_s - \frac{1}{2} \left( B_s (\tilde{B}_t - \tilde{B}_s) - \tilde{B}_s (B_t - B_s) \right). \quad (1)$$

**Theorem 1 (Optimal regularity and integrability of Lévy's Area)** *Set*

$$\begin{aligned} V_{\psi \circ \sqrt{\cdot}\text{-var};[0,1]}(A) &\equiv \sup_{(t_i) \subset [0,1]} \sum_i \psi \left( |A_{t_i, t_{i+1}}|^{1/2} \right), \\ |A|_{\psi \circ \sqrt{\cdot}\text{-var};[0,1]} &\equiv \inf \{ \varepsilon > 0 : V_{\psi \circ \sqrt{\cdot}\text{-var};[0,1]}(A/\varepsilon^2) \leq 1 \}. \end{aligned}$$

*Let  $\psi : [0, \infty) \rightarrow [0, \infty)$ ,  $\psi(0) = 0$  be continuous, strictly increasing, onto and such that  $\psi(x) = x^2 / \log \log(1/x)$  near 0+. Then  $|A|_{\psi \circ \sqrt{\cdot}\text{-var};[0,1]} < \infty$  a.s. and has a Gauss tail. Moreover, this  $\psi$ -variation is optimal in the sense that for any  $\tilde{\psi} : [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{x \rightarrow 0} \tilde{\psi}(x) / \psi(x) = \infty$  we have almost surely*

$$V_{\tilde{\psi} \circ \sqrt{\cdot}\text{-var};[0,1]}(A) = |A|_{\tilde{\psi} \circ \sqrt{\cdot}\text{-var};[0,1]} = +\infty.$$

**Remark 2 (Warning)** *We are not saying that Lévy's area, say  $|A_{0,1}|$ , has a Gauss tail. What we are saying is that quantities of type  $|A_{0,1}|^{1/2}$  have a Gauss tail. The square-root arises naturally if one seeks homogenous path space norm which deal simultaneously with paths and area and are crucial for our application in rough path theory.*

For example, theorem 1 sharpens the well-known statement [13] that for  $p > 2$  (which corresponds to  $\psi(x) = x^p$  above)

$$|A|_{p \circ \sqrt{\cdot}\text{-var};[0,1]} = \left( \sup_{D \subset [0,1]} \sum_{i: t_i \in D} |A_{t_i, t_{i+1}}|^{p/2} \right)^{1/p} < \infty \text{ a.s.}$$

(This variational regularity of Lévy area increments is precisely what allows to use rough path analysis in conjunction with Brownian motion.)

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<sup>3</sup> $(A_{0,t} : t \geq 0)$  is a continuous martingale, hence a time-change of Brownian motion. Conclude with Taylor's variation regularity.

As further application (and with regard to item (iii) above) rough path estimates are compatible with  $\psi$ -variation. For instance, Brownian motion and Lévy area can be enhanced with any number of higher iterated Stratonovich integrals and the resulting process has finite  $\psi$ -variation with Gaussian integrability of the associated homogenous norms, discussed in section 5.2. Similar arguments apply to sample path regularity of solution to stochastic differential equations in the rough path sense. When these are semimartingales (as is usually the case in Stratonovich theory) a.s. finite  $\psi$ -variation is seen just as for  $t \mapsto A_{0,t}$  above. On the other hand, our results imply that such regularity results are "robust" beyond semimartingales and apply to large classes of differential equations driven by Gaussian signals (including but far from restricted to fractional Brownian motion).

## 2 A Generalized Fernique Theorem

Let  $(\mathbb{B}, |\cdot|)$  be a real, separable Banach space equipped with its Borel  $\sigma$ -algebra  $\mathfrak{B}$  and a centered Gaussian measure  $\mu$ . A famous result by X. Fernique states that  $|\cdot|_* \mu$  has a Gauss tail; more precisely,

$$\int \exp(\eta |x|^2) d\mu(x) < \infty \text{ if } \eta < \frac{1}{2\sigma^2},$$

where

$$\sigma := \sup_{\xi \in B^*, |\xi|_{B^*} = 1} \left( \int \langle \xi, x \rangle^2 d\mu(x) \right)^{1/2} < \infty, \quad (2)$$

and this condition on  $\eta$  is sharp. See [10, Thm 4.1] for instance. We recall the notion of a *reproducing kernel Hilbert space*  $\mathcal{H}$ , continuously embedded in  $\mathbb{B}$ ,

$$|h| \leq \sigma |h|_{\mathcal{H}} \quad \forall h \in \mathcal{H},$$

so that  $(\mathbb{B}, \mathcal{H}, \mu)$  is an abstract Wiener space in the sense of L. Gross. (The standard example to have in mind is the Wiener-space  $C_0([0, 1], \mathbb{R})$  equipped with Wiener measure; then  $\mathcal{H}$  is the space of all absolutely continuous paths with  $h(0) = 0$  and  $\dot{h} \in L^2([0, 1])$ .) We can then cite Borell's inequality, e.g. [10, Theorem 4.3].

**Theorem 3** *Let  $(\mathbb{B}, \mathcal{H}, \mu)$  be an abstract Wiener space and  $A \subset E$  a measurable Borel set with  $\mu(A) > 0$ . Take  $a \in (-\infty, \infty]$  such that*

$$\mu(A) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx =: \Phi(a).$$

*Then, if  $\mathcal{K}$  denotes the unit ball in  $\mathcal{H}$  and  $\mu_*$  stands for the inner measure<sup>4</sup> associated to  $\mu$ ,*

$$\mu_*(A + r\mathcal{K}) = \mu_*\{x + rh : x \in A, h \in \mathcal{K}\} \geq \Phi(a + r). \quad (3)$$

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<sup>4</sup>Measurability of the so-called Minkowski sum  $A + r\mathcal{K}$  is a delicate topic. Use of the inner measure bypasses this issue and is not restrictive in applications.

The reader should observe that the following theorem reduces to the usual Fernique result when applied to the Banach norm on  $\mathbb{B}$ .

**Theorem 4 (Generalized Fernique Estimate)** *Let  $(\mathbb{B}, \mathcal{H}, \mu)$  be an abstract Wiener space. Assume  $f : \mathbb{B} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$  is a measurable map and  $N \subset \mathbb{B}$  a null-set such that for all  $b \notin N$*

$$|f(b)| < \infty \quad (4)$$

and for some positive constant  $c$ ,

$$\forall h \in \mathcal{H}: |f(b)| \leq c \{ |f(b-h)| + \sigma |h|_{\mathcal{H}} \}. \quad (5)$$

Then, with the definition of  $\sigma$  given in (2),

$$\int \exp \left( \eta |f(b)|^2 \right) d\mu(b) < \infty \quad \text{if } \eta < \frac{1}{2c^2\sigma^2}.$$

**Proof.** We have for all  $b \notin N$  and all  $h \in r\mathcal{K}$ , where  $\mathcal{K}$  denotes the unit ball of  $\mathcal{H}$  and  $r > 0$ ,

$$\begin{aligned} \{b : |f(b)| \leq M\} &\supset \{b : c(|f(b-h)| + \sigma |h|_{\mathcal{H}}) \leq M\} \\ &\supset \{b : c(|f(b-h)| + \sigma r) \leq M\} \\ &= \{b+h : |f(b)| \leq M/c - \sigma r\}. \end{aligned}$$

Since  $h \in r\mathcal{K}$  was arbitrary,

$$\begin{aligned} \{b : |f(b)| \leq M\} &\supset \cup_{h \in r\mathcal{K}} \{b+h : |f(b)| \leq M/c - \sigma r\} \\ &= \{b : |f(b)| \leq M/c - \sigma r\} + r\mathcal{K} \end{aligned}$$

and we see that

$$\begin{aligned} \mu[|f(b)| \leq M] &= \mu_*[|f(b)| \leq M] \\ &\geq \mu_* (\{b : |f(b)| \leq M/c - \sigma r\} + r\mathcal{K}) \end{aligned}$$

We can take  $M = (1 + \varepsilon) c \sigma r$  and obtain

$$\mu[|f(b)| \leq (1 + \varepsilon) c \sigma r] \geq \mu_* (\{b : |f(b)| \leq \varepsilon \sigma r\} + r\mathcal{K})$$

Keeping  $\varepsilon$  fixed, take  $r \geq r_0$  where  $r_0$  is chosen large enough such that

$$\mu[\{b : |f(b)| \leq \varepsilon \sigma r_0\}] > 0.$$

Letting  $\Phi$  denote the distribution function of a standard Gaussian, it follows from Borell's inequality that

$$\mu[|f(b)| \leq (1 + \varepsilon) c \sigma r] \geq \Phi(a + r)$$

for some  $a > -\infty$ . Equivalently,

$$\mu[|f(b)| \geq x] \leq \bar{\Phi}\left(a + \frac{x}{(1+\varepsilon)c\sigma}\right)$$

with  $\bar{\Phi} \equiv 1 - \Phi$  and using  $\bar{\Phi}(z) \lesssim \exp(-z^2/2)$  this we see that this implies

$$\int \exp\left(\eta |f(b)|^2\right) d\mu(b) < \infty$$

provided

$$\eta < \frac{1}{2} \left( \frac{1}{(1+\varepsilon)c\sigma} \right)^2.$$

Sending  $\varepsilon \rightarrow 0$  finishes the proof. ■

### 3 Regularity of stochastic processes

Sharp sample path properties for stochastic processes often require generalized Hölder- or variation norms. Using the following definition, Lévy's modulus for Brownian motion is captured by  $\varphi_{2,1}$ -Hölder regularity, Taylor's variation regularity corresponds to generalized  $\psi_{2,2}$ -variation. (Granted continuity and strictly monotonicity of  $\varphi$  and  $\psi$ , only the behaviour near zero matters.)

**Definition 5** *Given  $x > 0$  we define<sup>5</sup>*

$$\begin{aligned} \varphi_{p,1}(x) &= x^{1/p} \sqrt{\log_1 x} \text{ and } \psi_{p,1}(x) = \left| \frac{x}{\sqrt{\log_1 x}} \right|^p, \\ \text{where } \log_1(x) &= \begin{cases} \log \frac{1}{x} & , \text{ for } x \leq e^{-1} \\ 1 & , \text{ otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \varphi_{p,2}(x) &= x^{1/p} \sqrt{\log_2 x} \text{ and } \psi_{p,2}(x) = \left| \frac{x}{\sqrt{\log_2 x}} \right|^p, \\ \text{where } \log_2(x) &= \begin{cases} \log \log \frac{1}{x} & , \text{ for } x \leq e^{-e} \\ 1 & , \text{ otherwise} \end{cases} \end{aligned}$$

**Remark 6** *Note that  $\varphi_{p,2}(\psi_{p,2}(s)) \sim \psi_{p,2}(\varphi_{p,2}(s)) \sim s$  as  $s \rightarrow 0$ .*

We shall see that (sharp) generalized Hölder- or variation regularity of a stochastic process can be shown from the following simple condition. It is not only satisfied by a generic class of Gaussian processes and Gaussian rough paths (discussed in sections 4.1, 4.2 below) but also by Markov processes with uniform (sub)elliptic generator in divergence form [17, 16].

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<sup>5</sup>All  $\psi$ 's and  $\varphi$ 's below extend continuously to 0 with  $\psi(0) = 0$ ,  $\varphi(0) = 0$ .

**Condition 7**  $X$  is a process on  $[0, 1]$  taking values in a metric space  $(E, d)$  and there exists a  $\eta > 0$  s.t.

$$\sup_{0 \leq s < t \leq 1} \mathbb{E} \left( \exp \left( \eta \left[ \frac{d(X_s, X_t)}{|t-s|^{1/p}} \right]^2 \right) \right) < \infty, \quad (6)$$

Clearly, this condition guarantees the existence of a continuous version of  $X$  with which we always work.

**Lemma 8** Condition (6) is equivalent to

$$\sup_{0 \leq s < t \leq 1} \left| \frac{d(X_s, X_t)}{|t-s|^{1/p}} \right|_{L^{2q}(\mathbb{P})} = O(\sqrt{q}) \text{ as } q \rightarrow \infty.$$

**Proof.** Left to the reader. ■

### 3.1 Lévy's Modulus

Given  $x : [0, 1] \rightarrow (E, d)$  and  $\varphi : [0, \infty) \rightarrow [0, \infty)$ , strictly increasing,  $\varphi(x) = 0$  iff  $x = 0$ , we set

$$|x|_{\varphi\text{-Hö};[0,1]} := \sup_{0 \leq s < t \leq 1} \frac{d(x_s, x_t)}{\varphi(t-s)}.$$

**Theorem 9** Let  $X$  satisfy Condition 7. Assume  $\varphi \sim \varphi_{p,1}$  near 0+. Then the random variable

$$|X(\omega)|_{\varphi_{p,1}\text{-Hö};[0,1]}$$

has a Gauss-tail.

**Proof.** This is a straight-forward adaption of the case  $p = 2$  in [4]. We include details as we want to make the point that there is no obvious extension of these ideas to generalized variation "norms". The proof is based on the well-known Garsia-Rodemich-Rumsey lemma with the pair of functions  $\psi, q$  given by

$$\psi(x) := e^{\eta x^2} - 1, q(x) := x^{1/p}.$$

Setting  $\zeta(x) = \int_0^x u^{1/p-1} \sqrt{\log(1+1/u^2)} du$  this yields an estimate of form

$$\begin{aligned} d(X_s, X_t) &\leq c_1 \int_0^{t-s} u^{1/p-1} \sqrt{\log(1+4F/u^2)} du \\ &= c_1 F^{1/(2p)} \zeta\left(\frac{t-s}{\sqrt{F}}\right) \end{aligned}$$

where  $\zeta(x) \sim_{x \rightarrow 0} c_2 x^{1/p} \sqrt{\log 1/x} = c_2 \varphi_{p,1}(x)$  for  $x$  near 0+ and  $F = F(\omega)$  is given by

$$F = \iint_{[0,1]^2} \psi\left(\frac{d(X_u, X_v)}{q(|v-u|)}\right) du dv + 1 \quad (7)$$

By Condition 7 and Fubini,  $F \in L^1(\mathbb{P})$ ; adding 1 guarantees that  $F \geq 1$  which will be convenient below. By Fubini and Condition 7 we immediately see that  $F \in L^1$  (especially finite a.s.). An elementary computation reveals

$$\exists c_2 : \forall x, y \in [0, 1] : \zeta(xy) \leq c_2 \zeta(x) \zeta(y). \quad (8)$$

Combined with (7) we see that

$$d(X_s, X_t) \leq c_3 F^{1/(2p)} \zeta\left(1/\sqrt{F}\right) \zeta(t-s). \quad (9)$$

It remains to see that  $M(\omega) = M := F^{1/(2p)} \zeta\left(1/\sqrt{F}\right)$  has a Gauss tail. After a change of variables ( $\tilde{u} = F^{1/2}u$ )  $M = \int_0^1 \sqrt{\log(1 + F/u^2)} du$  and Jensen's inequality gives

$$\begin{aligned} \mathbb{E}[\exp(\lambda M^2)] &\leq \mathbb{E}\left[\int_0^1 \exp[\lambda \log(1 + F/u^2)] du\right] \\ &\leq \mathbb{E}\left[\int_0^1 \exp[\lambda \log(2F/u^2)] du\right] \\ &= (2F)^\lambda \mathbb{E}\left[\int_0^1 1/u^{2\lambda} du\right]. \end{aligned} \quad (10)$$

It now suffices to choose  $\lambda \in (0, 1/2)$ , so that the deterministic integral is finite, and to observe that  $1 \leq F^\lambda \leq F \in L^1(\mathbb{P})$ . To see that Gauss tail of  $\varphi_{p,1}$ -Hölder "norm", we split up the sup. For deterministic  $\delta$ , small enough, we have

$$\begin{aligned} \sup_{0 \leq s < t \leq 1} \frac{d(X_s, X_t)}{\varphi_{p,1}(|t-s|)} &\leq \sup_{\substack{0 \leq s < t \leq 1 \\ |t-s| \leq \delta}} \frac{d(X_s, X_t)}{\zeta(|t-s|)} \frac{\zeta(t-s)}{\varphi_{p,1}(|t-s|)} + \sup_{\substack{0 \leq s < t \leq 1 \\ |t-s| > \delta}} \frac{d(X_s, X_t)}{\varphi_{p,1}(|t-s|)} \\ &\leq c_4 M + c_5 |X|_{0,[0,1]}. \end{aligned}$$

Using Gaussian integrability of  $M$  and  $|X|_{0,[0,1]}$ , cf. Theorem 10, we see that  $|X|_{\varphi_{p,1}\text{-Hö},[0,1]}$  has a Gauss tail. The same argument works for any other  $\varphi$ -modulus for which  $\varphi \sim \varphi_{p,1}$  near 0+. th and in combination with (9) and (10) the last expression is the sum of two r.v. with a Gauss tail. It is easy to see that the sum of two r.v. with a Gauss tail has again a Gauss tail. The proof is finished by the observation that for every  $\varphi$  with  $\varphi(x) \sim \varphi_{p,1}(x)$  for  $x \rightarrow 0$  the same argument works. ■

**Theorem 10** *Let  $X$  satisfy Condition 7. Then there exists a constant  $C$  such that for all  $a < b$  in  $[0, 1]$  we have*

$$\mathbb{P}\left[|X|_{0,[a,b]} > x\right] \leq C \exp\left(-\frac{1}{C} \left(\frac{x}{|b-a|^{1/p}}\right)^2\right)$$

where  $|X|_{0,[a,b]} := \sup_{a \leq s < t \leq b} d(X_s, X_t)$ .



**Proof.**

$$\sup_{a \leq s < t \leq b} \mathbb{E} \exp \left( \eta \left( \frac{d(X_s, X_t)}{|t-s|^{1/p}} \right)^2 \right) < \infty$$

implies, setting  $Z_t = X_{a+t(b-a)}$ ,

$$\sup_{s, t \in [0, 1]} \mathbb{E} \exp \left( \eta \left( \frac{d(Z_s, Z_t)}{|b-a|^{1/p} |t-s|^{1/p}} \right)^2 \right) < \infty.$$

We have  $(b-a)^\alpha |X|_{\alpha\text{-H\"ol}; [a, b]} = |Z|_{\alpha\text{-H\"ol}; [0, 1]}$  and by Garsia-Rodemich-Rumsey, for any  $0 \leq \alpha < 1/p$ ,

$$\exists \tilde{\eta} > 0 : \mathbb{E} \exp \left( \tilde{\eta} \left( \frac{|Z|_{\alpha\text{-H\"ol}; [0, 1]}}{|b-a|^{1/p}} \right)^2 \right) < \infty.$$

It now suffices to take  $\alpha = 0$  and use Markov's inequality. ■

### 3.2 Taylor's Variation

**Theorem 11** *Let  $X$  satisfy Condition 7. Then with probability 1,*

$$V_{\psi_{p,2}}(X) := \sup_{D \subset [0,1]} \sum_{i: t_i \in D} \psi_{p,2}(d(X_{t_i}, X_{t_{i+1}})) < \infty. \quad (11)$$

*In the notation of appendix A this is equivalent to*

$$|X|_{\psi_{p,2}\text{-var}; [0,1]} < \infty \text{ a.s.}$$

Let us remark that (11) holds for any function  $\psi$  (in a reasonable class, cf. appendix A) for which

$$\limsup_{s \rightarrow 0} \psi(s) / \psi_{p,2}(s) < \infty;$$

this follows readily from (11)  $\Leftrightarrow$  (13). Good examples include  $s \mapsto s^{p+\varepsilon}$  and  $s \mapsto \psi_{p,1}(s)$ . It should be emphasized that the latter statement  $|X|_{\psi_{p,1}\text{-var}; [0,1]} < \infty$  a.s. is a trivial consequence of  $\varphi_{p,1}$ -H\"older regularity (cf. Theorem 9). However, examples show that finite  $\psi_{p,2}$ -variation can hold without having finite  $\varphi_{p,2}$ -H\"older modulus (e.g. Brownian motion with  $p = 2$ ).

**Lemma 12** *Consider a sequence of positive real numbers  $(h_n) \downarrow 0$  such that<sup>6</sup>*

$$C := 2 \sup_n h_{n-1}/h_n < \infty$$

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<sup>6</sup>This rules out  $e^{-n^2}$  for instance.

and define for each  $n \in \{1, 2, \dots\}$  a family of intervals

$$\mathcal{J}_n := \left\{ J_{n,i} := \left[ \frac{i}{2} h_n, \left( \frac{i}{2} + 1 \right) h_n \right] : i \in \{1, \dots, [2/h_n] + 1\} \right\}$$

Then there exists a  $\delta > 0$  only depending on  $(h_n)$  such that any interval  $(s, t) \subset [0, 1]$  with  $|t - s| < \delta$  is well approximated by some interval  $J_{n,i}$  in the sense that the following conditions are satisfied,

$$(t - s) \subset J_{n,i} \text{ and } |J_{n,i}| = h_n \leq C |t - s|.$$

Furthermore, for fixed  $(h_n)$  the choice of  $n$  depends only on  $(t - s)$  and  $n \uparrow \infty$  as  $(t - s) \downarrow 0$ .

**Proof.** We choose the largest  $n$  which still satisfies the second condition, that is

$$h_n \leq C(t - s) < h_{n-1}. \quad (12)$$

We then choose the largest possible index  $i$  so that  $J_{n,i}$  satisfies the first condition, that is

$$\frac{i}{2} h_n < s < \frac{i+1}{2} h_n,$$

and note that  $(s - \frac{i}{2} h_n) < h_n/2$ . To see that  $(s, t)$  is indeed contained in  $J_{n,i}$  it is enough to check that

$$h_n/2 + (t - s) \leq |J_{n,i}| = h_n \text{ or equivalently } \frac{t - s}{h_n} \leq \frac{1}{2}.$$

But this is true by (12) and definition of  $C$  since

$$\frac{t - s}{h_n} = \frac{t - s}{h_{n-1}} \frac{h_{n-1}}{h_n} \leq \frac{t - s}{C(t - s)} \sup_n \frac{h_{n-1}}{h_n} = \frac{1}{2}.$$

■

**Proof of Theorem 11.** As in [18] it is enough to show that

$$\lim_{\delta \rightarrow 0} \sup_{D \in \mathcal{D}(\delta)} \sum_{i: t_i \in D} \psi_{p,2}(d(X_{t_i}, X_{t_{i+1}})) < \infty \text{ a.s.} \quad (13)$$

For a given  $D$  call an interval  $(t_{i-1}, t_i)$  in the dissection  $D$  *good* if  $\psi_{p,2}(d(X_{t_i}, X_{t_{i+1}})) < c_1(t_i - t_{i-1})$  for some deterministic  $c_1$  to be determined by equation (17) below. Call this interval *bad* otherwise. Clearly,

$$\begin{aligned} \sum_i \psi_{p,2}(|X|) &= \sum_{\text{good intervals } (t_{i-1}, t_i)} + \sum_{\text{bad intervals } (t_{i-1}, t_i)} \\ &\leq c_1 + \sum_{\text{bad intervals } (t_{i-1}, t_i)} \end{aligned} \quad (14)$$

and we only need to deal with bad intervals. We will see that, provided  $|D| < \delta$  is small enough, the sum over the bad intervals can be controlled. Let  $(s, t)$  be

a bad interval in  $D$  with  $|t - s| < \delta_1$  where  $\delta_1$  is the constant whose existence is guaranteed by the previous lemma. The same lemma, applied with  $h_n = e^{-n}$  and  $C = 2e$ , implies that we can find  $i$  and  $n$  such that

$$(s, t) \subset J_{n,i} = \left[ \frac{i}{2} h_n, \left( \frac{i}{2} + 1 \right) h_n \right]$$

and  $h_n = |J_{n,i}| < 2e(t - s)$ . In particular,

$$\psi_{p,2} \left( |X|_{0,J_{n,i}} \right) \geq \psi_{p,2} (d(X_s, X_t)) \geq c_1 (t - s) \geq c_2 h_n \quad (15)$$

where we set  $c_2 = c_1 / (2e)$  and used that  $(s, t)$  is bad. Recalling  $\varphi_{p,2}(\psi_{p,2}(s)) \sim s$  as  $s \rightarrow 0$  and  $\varphi_{p,2}(\psi_{p,2}(s)) = s$  for  $s \geq 1$  we obviously have  $\varphi_{p,2}(\psi_{p,2}(s)) \leq c_3 s$ . Hence, using in particular Theorem 10, and writing  $c_4$  for the constant whose existence it guarantees,

$$\begin{aligned} & \mathbb{P} \left[ \psi_{p,2} \left( |X|_{0,J_{n,i}} \right) > c_2 h_n \right] \\ & \leq \mathbb{P} \left[ c_3 |X|_{0,J_{n,i}} > \varphi_{p,2}(c_2 h_n) \right] \\ & \leq c_4 \exp \left[ -\frac{1}{c_4} \left( \frac{\varphi_{p,2}(c_2 h_n)}{c_3 h_n^{1/p}} \right)^2 \right] \\ & = c_4 \exp \left[ -\frac{1}{c_4} \left( \frac{c_2^{1/p}}{c_3} \right)^2 \log_2(c_2 h_n) \right]. \end{aligned} \quad (16)$$

We now choose  $c_1$  such that

$$\frac{1}{c_4} \left( \frac{c_1^{1/p}}{c_3 (2e)^{1/p}} \right)^2 = \frac{1}{c_4} \left( \frac{c_2^{1/p}}{c_3} \right)^2 = 5p \quad (17)$$

Note that for  $n$  greater than some  $n_1$  large enough,  $\log_2(c_2 h_n) = \log(-\log(c_2 h_n))$ , and (16) reads

$$c_4 \frac{1}{(-\log(c_2 h_n))^{5p}} \leq c_4 \frac{2^{5p}}{n^{5p}} \equiv c_5 \frac{1}{n^{5p}}$$

where the estimate holds true provided  $n \geq n_2$  large enough so that

$$-\log(c_2 h_n) = -\log c_2 + n \geq n/2. \quad (18)$$

We established that

$$\mathbb{P} \left[ \psi_{p,2} \left( |X|_{0,J_{n,i}} \right) > c_2 h_n \right] \leq c_5 n^{-5p} \text{ for } n \geq n_1 \vee n_2.$$

If  $Z_n = Z_n(\omega)$  denotes the number of intervals in  $\mathcal{J}_n$  which satisfy

$$\psi_{p,2} \left( |X|_{0,J_{n,i}} \right) > c_2 h_n \quad (19)$$

Since  $Z_n$  is the sum (over  $i = 1, \dots, [2/h_n] + 1$ ) of all indicator functions of the events  $\left\{ \psi_{p,2} \left( |X|_{0,J_{n,i}} \right) > c_2 h_n \right\}$ ,

$$\begin{aligned} \mathbb{E}(Z_n) &\leq |\mathcal{J}_n| \times c_5 n^{-5p} \\ &\leq c_5 (2/h_n + 1) n^{-5p} \\ &= c_6 h_n^{-1} n^{-5p} \end{aligned}$$

Introduce the event  $A_n = \{Z_n > n^{-2p} h_n^{-1}\}$ . Then

$$\mathbb{P}(A_n) \leq n^{2p} h_n \mathbb{E}(Z_n) = c_6 n^{-3p}$$

and  $\sum_n \mathbb{P}(A_n) < \infty$ , after all we have  $p \geq 1$  fixed. The Borel-Cantelli lemma now implies that  $\mathbb{P}(A_n \text{ infinitely often}) = 0$ . Equivalently, with probability 1 there exists  $N(\omega)$  such that

$$Z_n \leq n^{-2p} h_n^{-1} \text{ for all } n \geq N(\omega).$$

Using a.s. finiteness of the Lévy's modulus "norm", theorem 9, there exists  $C_6(\omega)$ , finite almost surely, so that for any  $i \in \{1, \dots, [2/h_n] + 1\}$ ,

$$|X|_{0,J_{n,i}} < C_6 h_n^{1/p} \sqrt{\log_1 h_n}$$

From our definition of  $\psi_{p,2}$  we have  $\psi_{p,2}(s) \leq s^p$  for all  $s$  and so

$$\psi_{p,2} \left( |X|_{0,J_{n,i}} \right) \leq C_6^p h_n (\log_1 h_n)^{p/2}.$$

Now, for  $n \geq N(\omega)$ , the sum of  $\psi_{p,2} \left( |X|_{0,J_{n,i}} \right)$  over all intervals  $J_{n,i} \in \mathcal{J}_n$  which satisfy (19) is at most

$$Z_n \times C_6^p h_n (\log_1 h_n)^{p/2} \leq C_6^p n^{-2p} (\log_1 h_n)^{p/2} = C_6^p n^{-3p/2}. \quad (20)$$

As remarked in (15) every bad interval  $(s, t)$  of length smaller than  $\delta_1$  is contained in some  $J_{n,i} \in \mathcal{J}_n$  and such that  $\psi_{p,2} \left( |X|_{0,J_{n,i}} \right) \geq c_2 h_n$ . Let  $\delta = \delta(\omega) \in (0, \delta_1)$  be small enough such that the (in the sense of lemma above)  $n = n(\delta) > N(\omega)$  and so we are only dealing with intervals  $J_{n,i}$  to which our estimates apply. Then for any partition  $D$  with  $|D| < \delta(\omega)$ ,

$$\begin{aligned} &\sum_{\text{bad intervals } (t_{i-1}, t_i) \in D} \psi_{p,2} \left( d(X_{t_i}, X_{t_{i+1}}) \right) \\ &\leq \sum_{n=m(\delta(\omega))}^{\infty} \sum_{\substack{J_{n,i} \in \mathcal{J}_n \text{ for which} \\ (19) \text{ holds}}} \psi_{p,2} \left( |X|_{0,J_{n,i}} \right) \\ &\leq \sum_{n=m(\delta(\omega))}^{\infty} C_6^p n^{-3p/2} \quad \text{thanks to (20).} \end{aligned}$$

and this sum is finite almost surely as required. (The last step actually shows that we get a deterministic upper bound in (13) but this is irrelevant for our purposes.) ■

### 3.3 Law of Iterated Logarithm

A law of iterated logarithm also holds in the generality of the present setup of continuous processes on  $[0, 1]$  with values in some metric space.

**Proposition 13** *Let  $X$  satisfy Condition 7. Then there exists a constant  $C < \infty$  s.t.*

$$\limsup_{h \downarrow 0} \frac{|X|_{0;[0,h]}}{\varphi_{p,2}(h)} \leq C \text{ a.s.}$$

**Proof.** The idea is to scale by a geometric sequence. Although not strictly necessary for the conclusion, we show that  $C = \sqrt{c_1}$  where  $c_1$  is the constant whose existence is guaranteed by Theorem 10. To this end, fix  $\varepsilon > 0$ ,  $q \in (0, 1)$  and set  $c_2 = \sqrt{(1 + \varepsilon) c_1}$ . Let

$$A_n = \left\{ |X|_{0;[0,q^n]} \geq c_2 \varphi_{p,2}(q^n) \right\}.$$

From Theorem 10 we see that for  $n$  large enough

$$\begin{aligned} \mathbb{P}(A_n) &= \mathbb{P}\left[|X|_{0;[0,q^n]} \geq c_2 \varphi_{p,2}(q^n)\right] \\ &\leq c_1 \exp\left(-\frac{1}{c_1} \left(\frac{c_2 \varphi_{p,2}(q^n)}{q^{n/p}}\right)^2\right) \\ &= c_1 \exp\left(-\frac{1}{c_1} c_2^2 \log_2 q^n\right) \\ &= c_1 (-n \log q)^{-c_2^2/c_1} \end{aligned}$$

This is summable in  $n$  and hence, by the Borel-Cantelli lemma, we get that only finitely many of these events occur. It then follows easily that for all  $n \geq n_0(\varepsilon, \omega)$  and  $h$  small enough

$$q^{n+1} \leq h < q^n$$

and so, since  $\varphi_{p,2}(h)/h$  is decreasing,  $\varphi_{p,2}(q^n)/\varphi_{p,2}(q^{n+1}) \leq q^{-1}$ , and then

$$\frac{|X|_{0;[0,h]}}{\varphi_{p,2}(h)} \leq \frac{\varphi_{p,2}(q^n)}{\varphi_{p,2}(q^{n+1})} \frac{|X|_{0;[0,q^n]}}{\varphi_{p,2}(q^n)} \frac{\varphi_{p,2}(q^{n+1})}{\varphi_{p,2}(h)} \leq q^{-1} \sqrt{(1 + \varepsilon) c_1}.$$

We now pass to  $\limsup_{h \rightarrow 0}$ , followed by  $q \uparrow 1$  and  $\varepsilon \downarrow 0$ . This finishes the proof.  $\blacksquare$

## 4 Integrability of $\psi_{p,2}$ -variation norm

We have seen that a (continuous) process  $X : [0, 1] \rightarrow (E, d)$  which satisfies Condition 7 has a.s. finite  $\psi_{p,2}$ -variation. The aim of this section is to show that, for large classes of Gaussian processes and Gaussian rough paths, the  $\psi_{p,2}$ -variation "norm" (cf. appendix) enjoys Gaussian integrability. Due to Theorem 4 this reduces to check if the assumptions (4) and (5) of Theorem 4 are satisfied.

## 4.1 Gaussian paths

Assume that  $X$  is a centered (continuous) Gaussian process so that

$$\sup_D \sum_{i:(t_i) \in D} \left| \mathbb{E} \left( (X_{t_i, t_{i+1}})^2 \right) \right|^\rho < \infty$$

with sup taken over all dissections  $D$  of  $[0, 1]$ . This condition appears in [8] for instance. (For orientation, it holds for Brownian motion with  $\rho = 1$  and fractional Brownian motion with  $1/\rho = 1/(2H)$ ). Then a deterministic time-change of  $X$ , say  $Z$ , satisfies

$$\sup_{0 \leq s < t \leq 1} \frac{\mathbb{E} \left( |Z_{s,t}|^2 \right)}{|t - s|^{1/\rho}} < \infty.$$

With Gaussian integrability properties,  $\mathbb{E}(|Z_{s,t}|^p)^{1/p} \sim q^{1/2} \mathbb{E}(|Z_{s,t}|^2)^{1/2}$  this readily implies that  $Z$  satisfies condition 7. Using the invariance of (generalized) variation norms under reparametrization and theorem 11 we conclude that the sample paths of  $X$  are almost surely of finite  $\psi_{p,2}$ -variation for  $p = 2\rho$ . It is clear from the remark following theorem 11 that one also has finite  $V_{\psi;[0,1]}(x) < \infty$  for any other  $\psi$ -function such that  $\limsup_{s \rightarrow 0} \psi(s)/\psi_{p,2}(s) < \infty$ . If in addition to the standing assumptions (cf. appendix A) one has convexity of  $\psi$ , then  $x \mapsto |x|_{\psi\text{-var};[0,1]}$  gives rise to a Banach-norm  $x \mapsto |x|_{\psi\text{-var};[0,T]}$  and a Gauss tail follows from Fernique's classical result (or Theorem 4 applied to  $f = |\cdot|$ ).

## 4.2 Gaussian Rough paths

Fernique's classical estimates are not applicable to Gaussian rough paths since the homogenous norms involve the path level *and* Lévy area (which is not Gaussian). However, Gauss tail estimates for the  $\psi_{2,p}$ -variation norm (and any other  $\psi$  such that  $\psi \equiv \psi_{2,p}$  near 0+) do follow from our generalized Fernique theorem. We emphasize (again) that integrability properties of Wiener-Itô chaos (even Banach space valued) will not be sufficient for these estimates since the homogenous norms we are dealing with are fundamentally non-linear.

We shall need the result below.

**Theorem 14 ([6])** *Let  $X$  be a continuous, centered Gaussian process on  $[0, 1]$  with covariance  $R(s, t) = \mathbb{E}(X_s X_t)$  such that*

$$|R|_{\rho\text{-var},[0,1]^2} := \sup_{D, \tilde{D} \subset [0,1]} \left( \sum_{i,j:t_i \in D, \tilde{t}_j \in \tilde{D}} \left| \mathbb{E} \left[ X_{t_i, t_{i+1}} X_{\tilde{t}_j, \tilde{t}_{j+1}} \right] \right|^\rho \right)^{1/\rho} < \infty$$

*finite. Then, if  $\mathcal{H}$  denotes the Cameron-Martin space associated to  $X$ , we have the continuous embedding*

$$\mathcal{H} \hookrightarrow C^{\rho\text{-var}}([0, 1], \mathbb{R}^d).$$

More precisely, for all  $h \in \mathcal{H}$  and all  $0 \leq s < t \leq 1$ ,

$$|h|_{\rho\text{-var},[s,t]} \leq \sqrt{\langle h, h \rangle_{\mathcal{H}}} \sqrt{R_{\rho\text{-var},[s,t]^2}}.$$

We recall that a Lipschitz continuous path in  $\mathbb{R}^d$ , by simple computation of its area integral, lifts to a path in  $G^2(\mathbb{R}^d)$ , the step-2 nilpotent group with  $d$  generators equipped. We equip  $G^2(\mathbb{R}^d)$  with the Carnot-Caratheodory metric  $d$ ; this is natural, for instance the lifted path is then Lipschitz with respect to  $d$ . Hölder and variation regularity of paths  $[0, 1] \rightarrow (G^2(\mathbb{R}^d), d)$  are then special cases of the general discussion we had previously. We denote the resulting pathspace "norms" by  $\|\cdot\|_{\varphi\text{-Höl}}$  and  $\|\cdot\|_{\psi\text{-var}}$  etc. to distinguish from the general case.

**Theorem 15** *Let  $X$  be a centered, continuous Gaussian process in  $\mathbb{R}^d$  on  $[0, 1]$  with independent components. If the covariation of  $X$  is of finite  $\rho$ -variation for  $\rho < 3/2$  then there exists a lift to a Gaussian rough path  $\mathbf{X} \in C_0([0, 1], G^2(\mathbb{R}^d))$  of finite homogenous  $(2\rho + \varepsilon)$ -variation,  $\varepsilon > 0$  and  $\|\mathbf{X}\|_{\psi_{p,2}\text{-var};[0,1]}$  has a Gauss tail for  $p = 2\rho$ . More precisely, there exists  $\eta > 0$  such that*

$$\int \exp\left(\eta \|\mathbf{X}(\omega)\|_{\psi_{p,2}\text{-var};[0,1]}^2\right) d\mu(\omega) < \infty.$$

**Proof.** The first part of the statement (existence of lift) is proven in [6]. To show the Gauss tail we apply Theorem 4 with  $f = \|\cdot\|_{\psi_{p,2}\text{-var};[0,1]} \circ S_2$  where  $S_2 : X(\omega) = \omega \mapsto \mathbf{X}(\omega)$  denotes the lift constructed in [6]<sup>7</sup>, setting  $p = 2\rho$  we need to check (i)  $\|\mathbf{X}\|_{\psi_{p,2}\text{-var};[0,1]} < \infty$  a.s. and (ii)

$$\|\mathbf{X}\|_{\psi_{p,2}\text{-var};[0,1]} \leq c \left( \|T_{-h}\mathbf{X}\|_{\psi_{p,2}\text{-var};[0,1]} + \sigma |h|_{\mathcal{H}} \right). \quad (21)$$

Ad (i): We reparametrize (using the inverse of  $t \mapsto |R|_{\rho\text{-var};[0,t]^2}^\rho$ ) and obtain a continuous  $G^2(\mathbb{R}^d)$ -valued process  $\mathbf{Z}$  which satisfies ([6, Thm 35, equation (16)])

$$\|\mathbf{Z}_{s,t}\|_{L^q(\mathbb{P})} \leq C\sqrt{q}|t-s|^{\frac{1}{2\rho}}.$$

It now follows from theorem 11 that  $\mathbf{Z}$  has a.s. finite  $\psi_{p,2}$ -variation for  $p = 2\rho$  and thanks to invariance of generalized variation the same is true for  $\mathbf{X}$ .

Ad (ii). This is precisely theorem 28 below. We see in particular that any  $\eta < 1/(2c^2\sigma^2)$  where  $\sigma$  was defined in (2) and  $c$  is the constant which appears in (21) below. This finishes the proof. ■

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<sup>7</sup>A measurable map from

$$C_0([0, 1], \mathbb{R}^d) \rightarrow C_0([0, 1], G^2(\mathbb{R}^d)).$$

**Remark 16** *This theorem applies in particular to Brownian motion and fractional Brownian motion and their enhancement to rough paths (e.g. [2]), at least for Hurst parameter  $H > 1/3$ . (A Besov-variation embedding theorem [5] shows that  $H > 1/4$  is possible.) Using the well-known results of Taylor [18], Kawada-Kono [9] combined with the trivial  $|X|_{\psi_{p,2}\text{-var}} \leq \|\mathbf{X}\|_{\psi_{p,2}\text{-var}}$  we see from these examples that one cannot hope for stronger statements of this type.*

## 5 Applications

### 5.1 Regularity and Integrability of Lévy's Area

We now focus on the Lévy area  $A$  of  $d$ -dimensional Brownian motion  $B$ , defined by

$$A_{s,t} = \frac{1}{2} \left( \int_s^t (B_u - B_s) \otimes dB_u - \int_s^t (B_u - B_s) \otimes dB_u \right).$$

We already know that the Brownian motion enhanced with Lévy's area gives rise to a rough path of  $\psi_{2,2}$ -variation and the optimality of this is a consequence of the optimality of  $\psi_{2,2}$ -variation of the underlying Brownian motion alone. We now show that implicit variation regularity of Lévy's area is optimal in its own right.

**Proposition 17 (LIL for the Area process)** *There exists a positive constant  $C$ , such that*

$$\limsup_{h \rightarrow 0} \frac{\sqrt{|A_{0,h}|}}{\varphi_{2,2}(h)} \geq C \text{ a.s.}$$

*As a consequence, for every fixed  $t \in [0, 1)$ ,*

$$\limsup_{h \rightarrow 0} \frac{\sqrt{|A_{t,t+h}|}}{\varphi_{2,2}(h)} \geq C \text{ a.s.} \quad (22)$$

**Proof.** See [15, Theorem 2.15]. The consequence follows from  $A_{0,\cdot} \stackrel{\text{law}}{=} A_{t,t+\cdot}$  for fixed  $t$ . ■

**Theorem 18** *There exists a positive constant  $C$  such that*

$$\lim_{\delta \downarrow 0} \sup_{|D| < \delta} \sum_{i:t_i \in D} \psi_{2,2}(|A_{t_i,t_{i+1}}|^{1/2}) \geq C \text{ a.s.}$$

**Proof.** For brevity, we set  $\psi := \psi_{2,2}$ . It is convenient to define  $V_{\psi \circ \sqrt{\cdot}}^D(A) := \sum_{i:t_i \in D} \psi \circ \sqrt{\cdot}(|A_{t_i,t_{i+1}}|)$ . Then the proof works along the lines of [18]. We fix  $\varepsilon > 0$  and for every  $\delta > 0$  define

$$E_\delta := \left\{ t \in (0, 1) : \psi \left( |c^2 A_{t,t+h}|^{1/2} \right) > (1 - \varepsilon) h \text{ for some } h \in (0, \delta) \right\}.$$



where  $c$  is the constant which appears in (22). For fixed  $t$  and  $\delta$ ,  $\mathbb{P}(t \in E_\delta) = 1$  and by Fubini  $\mathbb{P}(|E_\delta| = 1) = 1$ . Now setting  $E := \cap_{\delta > 0} E_\delta = \cap_{n \geq 1} E_{\frac{1}{n}}$  gives

$$\mathbb{P}(|E| = 1) = 1.$$

This says that for every  $t \in E$  there exists an arbitrary small  $h > 0$ , such that

$$\psi\left(|c^2 A_{t,t+h}|^{1/2}\right) > (1 - \varepsilon)h$$

and these intervals of the form  $[t, t+h]$  form a Vitali covering of  $[0, 1]$ . Hence we can pick a finite disjoint union of such sets, each of length less than  $\delta$  but of total length of at least  $1 - \varepsilon$ . Now let  $D$  be a dissection with mesh  $|D| < \delta$  which includes the above collection as subintervals. Then

$$\begin{aligned} V_{\psi \circ \sqrt{\cdot}}^D(c^2 A) &\geq \sum_{i: t_i \in D} \psi\left(|c^2 A_{t_i, t_{i+1}}|^{1/2}\right) \\ &\geq \sum_{i: t_i \text{ is an element of the subcovering}} \psi\left(|c^2 A_{t_i, t_{i+1}}|^{1/2}\right) \\ &\geq (1 - \varepsilon) \sum_{i: t_i \text{ is an element of the subcovering}} h_i \\ &\geq (1 - \varepsilon)^2 \end{aligned}$$

Now we see for each  $\varepsilon > 0$  and  $\delta > 0$

$$P\left(\sup_{|D| < \delta} V_{\psi \circ \sqrt{\cdot}}^D(c^2 A) > (1 - \varepsilon)^2\right) = 1$$

Letting  $\varepsilon \rightarrow 0$  and then  $\delta \rightarrow 0$  (through a countable sequence) gives

$$\mathbb{P}\left(\lim_{\delta \downarrow 0} \sup_{|D| < \delta} V_{\psi \circ \sqrt{\cdot}}^D(c^2 A) \geq 1\right) = 1.$$

It is elementary to check from the definition of  $\psi$  that for all  $c \geq 0$  there exists a  $\Delta_c$  so that

$$\forall s \geq 0 : \psi(cs) \leq \Delta_c \psi(s)$$

This readily implies that

$$\mathbb{P}\left(\lim_{\delta \downarrow 0} \sup_{|D| < \delta} V_{\psi \circ \sqrt{\cdot}}^D(A) \geq \frac{1}{\Delta_c}\right) = 1$$

and the proof is finished. ■

As a corollary of this we can now give the proof of the regularity and integrability of Lévy's area as stated in theorem 1 in the introduction.

**Proof.** (Theorem 1) The Gauss tail of  $|A|_{|\psi\text{-var};[0,1]}$  is an obvious consequence of our general theorem 15 applied to  $d$ -dimensional Brownian motion. To see optimality take a  $\tilde{\psi}$  such that

$$\lim_{x \rightarrow 0} \frac{\tilde{\psi}(x)}{\psi_{p,2}(x)} = \infty. \quad (23)$$

Fix a dissection  $D = (t_i) \subset [0, 1]$  and write  $a_i = |A_{t_i, t_{i+1}}|^{1/2}$  for brevity. Then

$$\sum_i \psi_{p,2}(a_i) = \sum_i \tilde{\psi}(a_i) \frac{\psi_{p,2}(a_i)}{\tilde{\psi}(a_i)} \leq \left( \sup_i \frac{\psi_{p,2}(a_i)}{\tilde{\psi}(a_i)} \right) \sum_i \tilde{\psi}(a_i).$$

Thanks to the previous theorem, taking  $\lim_{\delta \downarrow 0} \sup_{|D| \leq \delta}$  leaves the left-hand-side above strictly positive. By a.s. (uniform) continuity of  $(s, t) \mapsto A_{s,t}$  we see that  $a_i$  will be arbitrarily small as  $|D| \downarrow 0$ , uniformly over all  $i$ . Thus,  $\sup_i \psi_{p,2}(a_i) / \tilde{\psi}(a_i) \rightarrow 0$  with  $|D| \downarrow 0$  and so we must indeed have that

$$\lim_{\delta \downarrow 0} \sup_{|D| \leq \delta} \sum_i \tilde{\psi}(a_i) = +\infty \text{ a.s.}$$

This finishes the proof of Theorem 1. ■

## 5.2 Regularity of Iterated Integrals, Stochastic Integrals and Solutions to stochastic differential equations in $\psi$ -variation

We now show that typical rough path estimates remain valid in  $\psi$ -variation.  $G^N(\mathbb{R}^d)$  denotes the free step- $N$  nilpotent group with  $d$ -generators and is the natural state space for rough paths. Equipped with Carnot-Carathéodory metric, it forms a metric space and the concept of  $\psi$ -variation is immediately meaningful. One of the basic theorems in rough path theory says that a continuous  $G^{[p]}(\mathbb{R}^d)$ -valued path  $\mathbf{x}(\cdot)$  of finite  $p$ -variation lifts uniquely to a  $G^N(\mathbb{R}^d)$ -valued path, say  $S_N(\mathbf{x})$ , also of finite  $p$ -variation, for any  $N \geq [p]$  and for all  $[s, t] \subset [0, 1]$

$$\|S_N(\mathbf{x})_{s,t}\| \leq \|S_N(\mathbf{x})\|_{p\text{-var};[s,t]} \leq C(N, p) \|\mathbf{x}\|_{p\text{-var};[s,t]}. \quad (24)$$

We now show how to extend this to  $\psi$ -variation. The proof is based on the clever concept of control functions introduced in the work of Lyons. These are continuous functions  $\omega : \{(s, t) : 0 \leq s \leq t \leq 1\} \rightarrow [0, \infty)$ , zero on the diagonal, and super-additive in the sense that  $\omega(s, t) + \omega(t, u) \leq \omega(s, u)$  for all  $0 \leq s \leq t \leq u \leq 1$ . The point is that<sup>8</sup>

$$d(x_s, x_t)^p \leq \omega(s, t) \iff |x|_{p\text{-var};[s,t]}^p \leq \omega(s, t).$$

Examples of control functions to have in mind are  $|t - s|^\theta$  for  $\theta \geq 1$  and  $|x|_{p\text{-var};[s,t]}^p$  as well as  $V_{\psi;[s,t]}(x)$ .

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<sup>8</sup>The concept makes sense for any continuous path with values in a metric space.

**Theorem 19** Let  $\mathbf{x} \in C^{\psi\text{-var}}([0, 1], G^{[p]}(\mathbb{R}^d))$  and  $\psi : [0, \infty) \rightarrow [0, \infty)$ ,  $\psi(0) = 0$ , continuous, strictly increasing and onto. Assume that  $\psi^{-1}(\cdot)^p$  is convex. Then, for  $N \geq [p]$  with  $C = C(N, p)$ ,

$$\|S_N(\mathbf{x})\|_{\psi\text{-var};[s,t]} \leq C \|\mathbf{x}\|_{\psi\text{-var};[s,t]}.$$

This estimate remains true with  $C = C(N, p, \varepsilon)$  if  $\psi^{-1}(\cdot)^p$  is only assumed convex on  $[0, \varepsilon]$  and  $\psi$  is such that  $\psi(\eta s) \leq \Delta_\eta \psi(s)$  for  $s$  near  $0+$  and  $\Delta_\eta \downarrow 0$  as  $\eta \downarrow 0$ . (These conditions are satisfied for any  $\psi \equiv \psi_{p,2}$  near  $0+$ .)

**Proof.** Step 1: We first assume that  $\psi^{-1}(\cdot)^p$  is convex on  $[0, \infty)$ . Fix  $[s, t] \subset [0, 1]$  and define for  $u, v \in [s, t]$ ,

$$\omega(u, v) := \sup_{D \subset [u, v]} \sum_{i: t_i \in D} \psi \left( \frac{\|x_{t_i, t_{i+1}}\|}{\|x\|_{\psi\text{-var};[s, t]}} \right).$$

Clearly,  $\omega$  is a control function on  $[s, t]$  and from the definition of  $\psi$ -variation it follows that  $\omega(s, t) \leq 1$ . It is also clear from the definition that

$$\psi \left( \frac{\|x_{u, v}\|}{\|x\|_{\psi\text{-var};[s, t]}} \right) \leq \omega(u, v)$$

and this can be rewritten as  $\|x_{u, v}\|^p \leq \|x\|_{\psi\text{-var};[s, t]}^p (\psi^{-1}(\omega(u, v)))^p$  and since the upper bound is a control function in  $(u, v)$  this yields

$$\|x\|_{p\text{-var};[u, v]}^p \leq \|x\|_{\psi\text{-var};[s, t]}^p (\psi^{-1}(\omega(u, v)))^p.$$

Using (24) (thanks to the convexity assumption, the right-hand-side is control function!) we get

$$\|S_N(x)_{u, v}\| \leq c_1 \|x\|_{\psi\text{-var};[s, t]} (\psi^{-1}(\omega(u, v))),$$

with  $c_1 = C(N, p)$ . Using this estimate and the superadditivity of  $\omega$  to see that

$$\begin{aligned} \sup_{D \subset [s, t]} \sum_{i: t_i \in D} \psi \left( \frac{\|S_N(x)_{t_i, t_{i+1}}\|}{c_1 \|x\|_{\psi\text{-var};[s, t]}} \right) &\leq \sup_{D \cap [s, t]} \sum_{i: t_i \in D} \omega(t_i, t_{i+1}) \\ &\leq \omega(s, t) \leq 1. \end{aligned}$$

By the definition of  $\psi$ -variation we conclude that  $\|S_N(x)\|_{\psi\text{-var};[s, t]} \leq c_1 \|x\|_{\psi\text{-var};[s, t]}$ .

Step-2: We now assume that  $\psi^{-1}(\cdot)^p$  is convex only on some interval  $[0, \varepsilon] \subset [0, 1]$ . Again, fix  $[s, t] \subset [0, 1]$  and define for  $u, v \in [s, t]$ ,

$$\omega(u, v) := \sup_{D \subset [u, v]} \sum_{i: t_i \in D} \psi \left( \frac{\|x_{t_i, t_{i+1}}\|}{M \|x\|_{\psi\text{-var};[s, t]}} \right)$$

where  $M$  is chosen large enough so that  $\omega(s, t) \leq \varepsilon$ . (This is possible by our assumption on  $\Delta_\eta$ .) Again,  $\omega$  is a control function on  $[s, t]$  and arguing exactly as in step 1, using that  $\psi^{-1}(\omega(u, v))^p$  is also a control on  $[s, t]$ , we led

$$\begin{aligned} \sup_{D \subset [s, t]} \sum_{i: t_i \in D} \psi \left( \frac{\|S_N(x)_{t_i, t_{i+1}}\|}{c_1 M \|x\|_{\psi\text{-var}, [s, t]}} \right) &\leq \sup_{D \cap [s, t]} \sum_{i: t_i \in D} \omega(t_i, t_{i+1}) \\ &\leq \omega(s, t) \leq \varepsilon \leq 1. \end{aligned}$$

By the definition of  $\psi$ -variation we conclude that  $\|S_N(x)\|_{\psi\text{-var}, [s, t]} \leq c_1 M \|x\|_{\psi\text{-var}, [s, t]}$ .  $\blacksquare$

**Corollary 20** *Let  $X$  be a Gaussian process satisfying the assumptions of Theorem 15 and let  $\mathbf{X}$  denote the corresponding rough path. Then  $S_N(\mathbf{X})$  has finite  $\psi_{p,2}$ -variation for every  $N > 2$  and  $\|S_N(\mathbf{X})\|_{\psi_{p,2}\text{-var}; [0,1]}$  has a Gauss tail. Applied to  $d$ -dimensional Brownian motion, this says that Brownian motion and all its iterated Stratonovich integrals up to order  $N$ , written as  $S_N(\mathbf{B})$  and viewed as a diffusion in the step- $N$  nilpotent group with  $d$  generators (e.g. [1, 12]), have Gaussian integrability in the sense that  $\|S_N(\mathbf{B})\|_{\psi_{2,2}\text{-var}; [0,1]} < \infty$  has a Gauss tail.*

**Remark 21** *To write this result in its most explicit form, take a multi-index  $I = (i_1, \dots, i_N) \in \{1, \dots, d\}^N$  and*

$$\begin{aligned} \Delta_{[t_i, t_{i+1}]}^N &\equiv \{(s_1, \dots, s_N) : t_i \leq s_1 < \dots < t_N < t_{i+1}\}, \\ \text{od} B^I &\equiv \text{od} B^{i_1} \circ \dots \circ \text{d} B^{i_N}. \end{aligned}$$

*The Gauss tail of  $\|S_N(\mathbf{B})\|_{\psi\text{-var}; [0,1]} < \infty$  is equivalent to the Gauss tail of*

$$\inf \left\{ \varepsilon > 0 : \sup_{D \subset [0,1]} \sum_{i: t_i \in D} \psi \left( \left| \frac{\int_{\Delta_{[t_i, t_{i+1}]}^N} \text{od} B^I}{\varepsilon^{|I|}} \right|^{\frac{1}{|I|}} \right) \leq 1 \right\}$$

*for all possible multi-indices  $I \in \{1, \dots, d\}^N$ .*

**Remark 22** (1) *The same argument shows that solutions to RDEs (rough differential equations) driven by some  $\mathbf{X}$  with finite  $\psi_{p,2}$ -variation also have finite  $\psi_{p,2}$ -variation (and the precise estimates from [7] also yields integrability properties for the corresponding  $\psi_{p,2}$ -seminorm).*

(2) *If one only considers the case of driving Brownian motion and the RDE solution as a path in Euclidean space (rather than a rough path in its own right), finite  $\psi_{2,2}$ -variation is easy to see directly: Under standard regularity assumptions, RDE solutions equal Stratonovich solutions and are seen to be continuous semimartingales. But every continuous local martingale is a time-change of Brownian motion (therefore of finite  $\psi_{2,2}$ -variation); the bounded variation part*

is of course harmless.

(3) The previous argument is not robust and relies crucially on the semimartingale nature of the RDE solution. There is a variety of examples of differential equations driven by Gaussian (and also Markovian) signals with non-semimartingale solutions. Our results imply in particular  $\psi_{p,2}$ -variation regularity of sample paths is a robust property and valid beyond the usual semimartingale context.

## A Appendix: Variation norms

The original definitions of [14] and [11] extend immediately to paths with values in a metric space  $(E, d)$ . The recent monograph [3] contains extensive references to  $\psi$ -variation.

**Definition 23** Let  $x \in C([0, 1], E)$  and  $\psi : [0, \infty) \rightarrow [0, \infty)$ ,  $\psi(0) = 0$ , continuous, strictly increasing and onto<sup>9</sup>. Then we define the  $\psi$ -variation "norm" of  $x$  over  $[s, t]$  by

$$|x|_{\psi\text{-var}, [s, t]} := \inf\{\varepsilon > 0 : \underbrace{\sup_{D \subset [s, t]} \sum_{i: t_i \in D} \psi\left(\frac{d(x_{t_i}, x_{t_{i+1}})}{\varepsilon}\right)}_{=: V_{\psi; [s, t]}^\varepsilon(x)} \leq 1\}.$$

We write  $V_{\psi, [s, t]}(x)$  rather than  $V_{\psi, [s, t]}^\varepsilon(x)$  when  $\varepsilon = 1$  and also define

$$C^{\psi\text{-var}}([0, 1], E) := \left\{x \in C([0, 1], E) : |x|_{\psi\text{-var}, [0, 1]} < \infty\right\}.$$

It may help to keep in mind that for  $\psi(s) = s^p$  this is consistent with

$$V_{(\cdot)^p; [s, t]}(x) \equiv |x|_{p\text{-var}, [s, t]}^p := \sup_{D \subset [s, t]} \sum_{i: t_i \in D} d(x_{t_i}, x_{t_{i+1}})^p.$$

**Definition 24** We say that a function  $\psi : [0, \infty) \rightarrow [0, \infty)$  has the property  $(d_2^{loc})$ , if

$$\limsup_{s \downarrow 0} \psi^{-1}(2\psi(s)) / s < \infty.$$

Equivalently, for any  $T > 0$  there exists a constant  $d_2 = d_2(T)$  such that  $\forall s \in [0, T] : 2\psi(s) \leq \psi(d_2 s)$ .

We say that a function  $\psi : [0, \infty) \rightarrow [0, \infty)$  has the property  $(\Delta_2^{loc})$ , if

$$\limsup_{s \downarrow 0} \psi(2\psi(s)) / s < \infty.$$

Equivalently, for any  $T > 0$  there exists a constant  $\Delta_2 = \Delta_2(T)$  such that  $\forall s \in [0, T] : \psi(2s) \leq \Delta_2 \psi(s)$ .

We speak of properties  $d_2$  and  $\Delta_2$  if the  $\limsup_{s \downarrow 0}$  can be replaced by  $\sup_{s \in \mathbb{R}^+}$ .

<sup>9</sup>These assumptions can be relaxed but allow us to consider  $\psi^{-1}$  without added technicalities.

Assuming the above properties, the notion of  $\psi$ -variation behaves "as expected".

**Proposition 25** *Let  $\psi : [0, \infty) \rightarrow [0, \infty)$ ,  $\psi(0) = 0$ , continuous, strictly increasing, onto and let  $x \in C([0, 1], E)$ . Assume further that  $\psi$  fulfills the properties  $(d_2^{loc})$  and  $(\Delta_2^{loc})$ . Then*

$$V_{\psi;[0,1]}(x) < \infty \Leftrightarrow |x|_{\psi\text{-var};[0,1]} < \infty.$$

**Proof.** Using  $(d_2^{loc})$  with the result constant  $d_2 = d_2(T)$  with  $T = 1$  say (for  $\varepsilon$  large enough),

$$\begin{aligned} 2 \lim_{\varepsilon \rightarrow \infty} V_{\psi;[0,1]}^\varepsilon(x) &\leq \lim_{\varepsilon \rightarrow \infty} V_{\psi;[0,1]}^{\varepsilon/d_2}(x) \\ &= \lim_{\varepsilon \rightarrow \infty} V_{\psi;[0,1]}^\varepsilon(x) \\ &\leq V_{\psi;[0,1]}(x) < \infty \end{aligned}$$

It follows that  $\lim_{\varepsilon \rightarrow \infty} V_{\psi;[0,1]}^\varepsilon(x) = 0$  and in particular there exists  $\varepsilon^*$  such that  $V_{\psi;[0,1]}^{\varepsilon^*}(x) \leq 1$ . To see the other direction, we make use of  $(\Delta_2^{loc})$ . Of course, we may assume  $|x|_{\psi\text{-var};[0,1]} > 0$  and note that  $(\Delta_2^{loc})$  implies for every  $c \geq 0$  and every  $T \geq 0$  the existence of a constant  $\Delta_c$  such that  $\psi(cs) \leq \Delta_c \psi(s)$  for all  $s \in [0, T]$ . Choosing  $T = \frac{|x|_{0,[0,1]}}{|x|_{\psi\text{-var};[0,1]}}$  we can estimate

$$\begin{aligned} \sum_i \psi(d(x_{t_i}, x_{t_{i+1}})) &\leq \sum_i \psi\left(\frac{d(x_{t_i}, x_{t_{i+1}})}{|x|_{\psi\text{-var};[0,1]}} |x|_{\psi\text{-var};[0,1]}\right) \\ &\leq \Delta_c \sum_i \psi\left(\frac{d(x_{t_i}, x_{t_{i+1}})}{|x|_{\psi\text{-var};[0,1]}}\right) \leq \Delta_c \end{aligned}$$

with  $c = |x|_{\psi\text{-var};[0,1]}$ . Taking the supremum over all dissections  $D = (t_i) \subset [0, 1]$  the result follows. ■

**Lemma 26** *Let  $\psi : [0, \infty) \rightarrow [0, \infty)$ ,  $\psi(0) = 0$ , continuous, strictly increasing, onto and let  $x \in C([0, 1], E)$ . Then, for all  $0 \leq s < t \leq 1$ ,*

$$|x_{s,t}| \leq \psi^{-1}(1) |x|_{\psi\text{-var};[s,t]}.$$

**Proof.** Clearly,

$$\frac{|x_{s,t}|}{|x|_{\psi\text{-var};[s,t]}} \leq \psi^{-1}\left(V_{\psi\text{-var};[s,t]}\left(\frac{x}{|x|_{\psi\text{-var};[s,t]}}\right)\right)$$

and by definition of  $V_\psi$  it follows that  $|x_{s,t}| \leq \psi^{-1}(1) |x|_{\psi\text{-var};[s,t]}$  as required. ■

## B Appendix: Rough Paths

### B.1 Notation

Complete expositions of rough path theory include [12], [13]. For the reader's convenience we review some basic concepts. For a fixed  $N \geq 0$ , the truncated tensor algebra  $T^N(\mathbb{R}^d)$  of degree  $N$  is defined as

$$T^N(\mathbb{R}^d) := \oplus_{k=0}^N (\mathbb{R}^d)^{\otimes k},$$

where we set  $(\mathbb{R}^d)^0 := \mathbb{R}^d$  and  $(\mathbb{R}^d)^{\otimes k}$ , equipped with the usual tensor product. Then  $T^N(\mathbb{R}^d)$  forms an algebra under the usual scalar product and vector addition. Using Young integrals we can define for a path  $x \in C^{p\text{-var}}([0, 1], \mathbb{R}^d)$  with  $p \in [1, 2)$  its step- $N$  lift by

$$\begin{aligned} S_N(x) : [0, 1] &\rightarrow T^N(\mathbb{R}^d) \\ t &\mapsto 1 + \sum_{i=i}^N \int_{0 < s_1 < \dots < s_k < t} dx \otimes \dots \otimes dx \end{aligned}$$

This yields a path  $S_N(x)$  with values in  $T^N(\mathbb{R}^d)$ . There are numerous non-linear relations between these iterated integrals and  $S_N(x)$  stays in a the proper subset

$$G^N(\mathbb{R}^d) := \{ \mathbf{g} \in T^N(\mathbb{R}^d) : \exists x \text{ smooth: } S_N(x)_1 = \mathbf{g} \}.$$

It turns out that  $G^N(\mathbb{R}^d)$  is a Lie group under induced tensor multiplication and (a possible realization of) the step- $N$  free nilpotent group with  $d$  generators. The Carnot-Caratheodory (CC) "norm"

$$\|\mathbf{g}\| = \inf \left\{ \int_0^1 |\dot{x}_u| du : x \text{ smooth, } S_N(x)_1 = \mathbf{g} \right\}$$

induces a (left-invariant) metric, the CC metric  $d : (\mathbf{g}, \mathbf{h}) \mapsto \|\mathbf{g}^{-1} \otimes \mathbf{h}\|$ . The CC norm is homogenous under dilation

$$\mathbf{g} = 1 + \mathbf{g}^1 + \dots + \mathbf{g}^N \mapsto 1 + \lambda \mathbf{g}^1 + \dots + \lambda^N \mathbf{g}^N.$$

and using equivalence of continuous, homogenous norms (by a compactness argument) one the Lipschitz equivalence

$$\|\mathbf{g}\| \sim |\mathbf{g}^1| + |\mathbf{g}^2|^{1/2} + \dots + |\mathbf{g}^N|^{1/N}$$

which is very useful in computations. Having a metric structure on  $G^N(\mathbb{R}^d)$  we can speak paths of finite  $p$ - or  $\psi$ -variation, as well as of  $\alpha$ -Hölder or  $\varphi$ -Hölder regularity. The class of (weak geometric)  $p$ -rough path is defined as

$$C^{p\text{-var}}([0, 1], G^{[p]}(\mathbb{R}^d)).$$

The order of nilpotency is tied to the regularity of the path. In the example of Brownian motion and Lévy's area (which give rise to a path in the step-2 group, one can take  $p \in (2, 3)$ ). Rough path theory implies that  $p$ -rough paths drive differential equations in a fully deterministic way with a variety of continuity properties of the solution map.

## B.2 Translating Rough paths

We are interested in the translates of rough paths by paths of a Cameron-Martin space. For Brownian motion it is obvious that Cameron-Martin paths are of finite 1-variation. On the other hand, we have seen that (theorem 14) for many Gaussian processes  $X$  one has  $\mathcal{H}(X) \hookrightarrow C^{\rho\text{-var}}$  for some  $\rho$  where  $\mathcal{H}(X)$  is the Cameron-Martin space associated to  $X$ . We recall the definition of the translation operator in rough path theory (cf. [13]).

**Definition 27** For  $\mathbf{x} \in C^{p\text{-var}}([0, 1], G^2(\mathbb{R}^d))$ ,  $p \geq 1$  and  $h \in C^{\rho\text{-var}}([0, 1], \mathbb{R}^d)$  such that  $1/p + 1/\rho > 1$ , we define the translation operator  $T_h$  by

$$T_h(\mathbf{x})_{s,t} = 1 + (\mathbf{x}_{s,t}^1 + h_{s,t}) + \left( \mathbf{x}_{s,t}^2 + \int_s^t x_{s,u} \otimes dh_u + \int_s^t h_{s,u} \otimes dx_u + \int_s^t h_{s,u} \otimes dh_u \right)$$

where all cross integrals are well-defined Young integrals.

One should note that for smooth paths  $x, h$  and  $\mathbf{x} = S_2(x)$ , the step-2 lift of  $x$ , the translate  $T_h(\mathbf{x})$  is precisely  $S_2(x + h)$ .

**Theorem 28** Assume  $p \geq 1$  and  $1/p + 1/\rho > 1$ . Let  $\psi : [0, \infty) \rightarrow [0, \infty)$ ,  $\psi(0) = 0$ , continuous, strictly increasing and onto. Assume that  $\psi$  satisfies the property  $(d_2^{\text{loc}})$  and that  $\psi$  is such that  $\psi(\eta s) \leq \Delta_\eta \psi(s)$  for every  $\eta > 0$ ,  $s$  near  $0+$  and that  $\Delta_\eta \downarrow 0$  as  $\eta \downarrow 0$ . Furthermore, assume

1. The function  $x \mapsto \psi(x^{1/\rho})$  is convex near  $0+$
2. The function  $x \mapsto (\psi^{-1}(x))^p$  is convex near  $0+$

(These conditions are satisfied for any  $\psi \equiv \psi_{p,2}$  near  $0+$ .) Then there exists a constant  $C > 0$  such that for all  $\mathbf{y} \in C^{\psi\text{-var}}([0, 1], G^2(\mathbb{R}^d))$  and  $h \in C^{\rho\text{-var}}([0, 1], \mathbb{R}^d)$

$$\|T_h(\mathbf{y})\|_{\psi\text{-var};[s,t]} \leq C \left( \|\mathbf{y}\|_{\psi\text{-var};[s,t]} + |h|_{\rho\text{-var};[s,t]} \right). \quad (25)$$

**Proof.** Note that (25) is equivalent (by homogeneity of the norm) to the statement that for all  $\lambda > 0$  small enough (depending on  $\mathbf{y}, h$  and  $\psi$ )

$$\|T_{\lambda h}(\delta_\lambda \mathbf{y})\|_{\psi\text{-var};[s,t]} \leq C \left( \|\delta_\lambda \mathbf{y}\|_{\psi\text{-var};[s,t]} + |\lambda h|_{\psi\text{-var};[s,t]} \right).$$

Therefore we can assume in the rest of the proof that the variation "norms" of  $y$  and  $h$  are arbitrary small (and therefore the increments as well, cf. Lemma



26). By equivalence of homogenous norms, the CC norm of the group element  $T_h(\mathbf{y})_{s,t}$  is estimated by

$$\begin{aligned} \|T_h(\mathbf{y})_{s,t}\| &\lesssim |h_{s,t} + \mathbf{y}_{s,t}^1| \\ &\quad + \sqrt{|\mathbf{y}_{s,t}^2|} + \sqrt{\left|\int_s^t h_{s,r} \otimes dy_r\right|} + \sqrt{\left|\int_s^t y_{s,r} \otimes dh_r\right|} + \sqrt{\left|\int_s^t h_{s,r} \otimes dh_r\right|}. \end{aligned}$$

By assumption  $\psi$  is "quasi-subadditive" in the sense that

$$\psi(s+t) \leq \psi(2 \max(s, t)) \leq \Delta_2 \psi(\max(s, t)) \leq \Delta_2 (\psi(s) + \psi(t)),$$

as long as  $s, t \in [0, 1]$ , say. Absorbing such constants into  $\lesssim$  we have

$$\begin{aligned} \psi(\|T_h \mathbf{y}_{s,t}\|) &\lesssim \left( \psi(\|\mathbf{y}_{s,t}\|) + \psi\left(\|S_2(h)_{s,t}\| \right) \right. \\ &\quad \left. + \psi\left(\sqrt{\left|\int_s^t h_{s,r} \otimes dy_r\right|}\right) + \psi\left(\sqrt{\left|\int_s^t y_{s,r} \otimes dh_r\right|}\right) \right). \end{aligned}$$

By Young's inequality and using  $\psi(cs) \leq \Delta_c \psi(s)$  we see  $\psi\left(\|S_2(h)_{s,t}\| \right) \lesssim \psi(|h|_{\rho\text{-var};[s,t]})$ . We now show that

$$\psi\left(\sqrt{\left|\int_s^t y_{s,r} \otimes dh_r\right|}\right) \leq V_{\psi;[s,t]}(y) + \psi(|h|_{\rho\text{-var};[s,t]}).$$

To this end, we note that  $|y_{s,t}| \leq \psi^{-1}(V_{\psi;[s,t]}(y))$  and since  $(\psi^{-1}(x))^p$  is convex near 0+ the right hand side of  $|y_{s,t}|^p \leq [\psi^{-1}(V_{\psi;[s,t]}(y))]^p$  is a control from which it follows that  $|y|_{p\text{-var};[s,t]} \leq [\psi^{-1}(V_{\psi;[s,t]}(y))]$  and we rewrite this as

$$\psi(|y|_{p\text{-var};[s,t]}) \leq (V_{\psi;[s,t]}(y)).$$

Using Young's inequality ( $1/p + 1/\rho > 1$ ), quasi-subadditivity, and the estimate of the previous line,

$$\begin{aligned} \psi\left(\sqrt{\left|\int_s^t y_{s,r} \otimes dh_r\right|}\right) &\leq \psi\left(c \sqrt{|y|_{p\text{-var};[s,t]} |h|_{\rho\text{-var};[s,t]}}\right) \\ &\lesssim \psi(|y|_{p\text{-var};[s,t]}) + \psi(|h|_{\rho\text{-var};[s,t]}) \\ &\leq V_{\psi;[s,t]}(y) + \psi(|h|_{\rho\text{-var};[s,t]}). \end{aligned} \tag{26}$$

What makes this estimate useful is that  $V_{\psi;[s,t]}$  and  $\psi(|h|_{\rho\text{-var};[s,t]})$ , thanks to assumption 1, are controls and hence super-additive in  $(s, t)$ . Replacing  $s, t$  by

some arbitrary interval in a dissection  $D$  of  $[s, t]$ , following by summation and taking  $\sup_{D \subset [s, t]}$  then gives

$$\begin{aligned} V_{\psi; [s, t]} \left( \sqrt{\left| \int y_{\cdot, r} \otimes dh_r \right|} \right) &\equiv \sup_{D \subset [s, t]} \sum_{i: t_i \in D} \psi \left( \sqrt{\left| \int_{t_i}^{t_{i+1}} y_{t_i, r} \otimes dh_r \right|} \right) \\ &\lesssim V_{\psi; [s, t]}(\mathbf{y}) + \psi \left( |h|_{\rho\text{-var}; [s, t]} \right). \end{aligned}$$

Of course, the same arguments applies to the other mixed integral,

$$V_{\psi; [s, t]} \left( \sqrt{\left| \int h_{\cdot, r} \otimes dy_r \right|} \right) \lesssim V_{\psi; [s, t]}(\mathbf{y}) + \psi \left( |h|_{\rho\text{-var}; [s, t]} \right).$$

Thus, we have shown so far that

$$V_{\psi; [s, t]}(T_h(\mathbf{y})) \lesssim V_{\psi; [s, t]}(\mathbf{y}) + \psi \left( |h|_{\rho\text{-var}; [s, t]} \right). \quad (27)$$

To prepare for the remainder of the argument consider  $a_\varepsilon, b_\varepsilon, c_\varepsilon$  all decreasing in  $\varepsilon \in (0, \infty)$  such that  $a_\varepsilon \leq b_\varepsilon + c_\varepsilon \forall \varepsilon$ . Then

$$\begin{aligned} \inf \{ \varepsilon > 0 : a_\varepsilon \leq 1 \} &\leq \inf \{ \varepsilon > 0 : b_\varepsilon + c_\varepsilon \leq 1 \} \\ &\leq \inf \{ \varepsilon > 0 : b_\varepsilon \leq 1/2 \} \\ &\quad \vee \inf \{ \varepsilon > 0 : c_\varepsilon \leq 1/2 \}. \end{aligned} \quad (28)$$

We now use (27), (28),

$$\begin{aligned} \|T_h(\mathbf{y})\|_{\psi\text{-var}; [s, t]} &\equiv \inf \left( \varepsilon > 0 : V_{\psi; [s, t]}(\delta_{1/\varepsilon}(T_h(\mathbf{y}))) \leq 1 \right) \\ &\leq \inf \left( \varepsilon > 0 : c_1 V_{\psi; [s, t]}(\delta_{1/\varepsilon}\mathbf{y}) + c_1 \psi \left( |h|_{\rho\text{-var}; [s, t]} / \varepsilon \right) \leq 1 \right) \\ &\leq \inf \left( \varepsilon > 0 : c_1 V_{\psi; [s, t]}(\delta_{1/\varepsilon}\mathbf{y}) \leq \frac{1}{2} \right) \\ &\quad \vee \inf \left( \varepsilon > 0 : c_1 \psi \left( |h|_{\rho\text{-var}; [s, t]} / \varepsilon \right) \leq \frac{1}{2} \right) \\ &= (I) + (II) \end{aligned}$$

To deal with (II), choose  $\varepsilon = c_2 |h|_{\rho\text{-var}; [s, t]}$  and then  $c_2$  large enough so that  $c_1 \psi(1/c_2) \leq 1/2$ . For this  $c_2$  we then have  $(II) \leq c_2 |h|_{\rho\text{-var}; [s, t]}$ . To deal with (I), we choose  $\varepsilon = \|\mathbf{y}\|_{\psi\text{-var}; [s, t]} / \eta$  and using the assumption on  $\Delta_\eta$

$$\begin{aligned} &\leq c_3 \inf \left( \varepsilon > 0 : V_{\psi; [s, t]}(\delta_{1/\varepsilon}\mathbf{y}) \leq 1 \right) \\ &\quad \vee c_2 |h|_{\rho\text{-var}; [s, t]} \\ &\lesssim \|\mathbf{y}\|_{\psi\text{-var}; [s, t]} + |h|_{\rho\text{-var}; [s, t]}. \end{aligned}$$

■

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